Bell Basis and Measurements

BELL BASIS

So far we have concentrated most of our efforts on the spin-1/2 singlet. However, this is only one of four states. In the polarization-spatial mode basis, there are four possible states

$$|H_1, H_2\rangle \tag{1}$$
$$|H_1, V_2\rangle \tag{2}$$
$$|V_1, H_2\rangle \tag{3}$$
$$|V_1, V_2\rangle \tag{4}$$
$$\tag{5}$$

We have already written the singlet state in a decomposition of these states namely

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|H_1, V_2\rangle - |V_1, H_2\rangle\right) \tag{6}$$

All four states are

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|H_1, V_2\rangle - |V_1, H_2\rangle\right) \tag{7}$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|H_1, V_2\rangle + |V_1, H_2\rangle\right) \tag{8}$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|H_1, H_2\rangle - |V_1, V_2\rangle\right) \tag{9}$$
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|H_1, H_2\rangle + |V_1, V_2\rangle\right) \tag{10}$$
$$\tag{11}$$

Prove that these states are orthogonal.

Now consider the effects of waveplates on these states. First consider that a quarter waveplate is in the path of particle 1. Also, the waveplate is set such that the fast axis is aligned with the vertical polarization axis and thus the slow axis is aligned with the horizontal polarization axis. We assume that the states after the waveplate are given by bell states above. Consider the transformation if the quarter waveplate is rotated by 90°. The horizontal component gets a phase boost of a quarter wave and the vertical component gets a phase lag of a quarter wave. The net effect is that the phase of state is inverted. We are thus left with the quarter waveplate performing the transformation

$$|\Psi^-\rangle \leftrightarrow |\Psi^+\rangle \tag{12}$$

$$|\Phi^-\rangle \leftrightarrow |\Phi^+\rangle \tag{13}$$

Next consider the effect of a half waveplate which undergoes rotations of 45°. We first assume that the states after the half waveplate are given by the above Bell states. If the half waveplate is then rotated by 45°, then the horizontal polarization is transformed to vertical polarization and vice versa. We are then left with

$$|\Psi^-\rangle \leftrightarrow |\Phi^+\rangle \tag{14}$$

$$|\Psi^+\rangle \leftrightarrow |\Phi^-\rangle \tag{15}$$

BELL MEASUREMENT

How does one measure the Bell states. Of course one can measure them in the original basis using polarizing beam splitters. However, to uniquely determine a state, one would need an ensemble identically prepared pairs. Therefore, we must analyze the Bell states using a different technique. The beam splitter can do this. However, the beam splitter has some drawbacks which we will discuss.

Recall that the beamsplitter transformation is given by

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \tag{16}$$

One should also keep in mind that if one is using a beam splitter and injecting the two photons with two unique spatial modes into a beam splitter then it is possible to have both photons leave the same output port. For example, consider the simple example of two photons entering different ports each with all other quantum features of the two photons being identical. The wavefunction

$$
\begin{aligned}
|\Psi\rangle &= a_1^\dagger a_2^\dagger |0\rangle & (17)\\
&\rightarrow \frac{1}{2}(a_1^\dagger + i a_2^\dagger)(i a_1^\dagger + a_2^\dagger)|0\rangle & (18)\\
&= \frac{1}{2}(i(a_1^\dagger)^2 i + (a_2^\dagger)^2)|0\rangle & (19)\\
&= \frac{1}{\sqrt{2}}\left(|2,0\rangle + |0,2\rangle\right) & (20)
\end{aligned}
$$

Thus, both photons leave the same output port. This type of interference is called the Hong-Ou-Mandel dip, because if one looks at the two-photon coincidences in different ports, there will be a dip in the two photon coincidences once all of the quantum features become indistinguishable due to the beam splitter. This example demonstrated the fact that the initial four dimensional state space is now enlarged by using a beam splitter.

Now consider the transformations of the $|\Psi^+\rangle$ and $|\Psi^-\rangle$ Bell state at a beamsplitter. The transformations are

$$
\begin{aligned}
|\Psi^\pm\rangle &= \frac{1}{\sqrt{2}}\left(a_h^\dagger b_v^\dagger \pm a_v^\dagger b_h^\dagger\right)|0\rangle & (21)\\
&\rightarrow \frac{1}{2\sqrt{2}}\left((a_h^\dagger + i b_h^\dagger)(i a_v^\dagger + b_v^\dagger) \pm (a_v^\dagger + i b_v^\dagger)(i a_h^\dagger + b_h j^\dagger)\right)|0\rangle & (22)
\end{aligned}
$$

which yields

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}|H_1, V_1\rangle + |V_2, H_2\rangle \tag{23}$$

and which yields

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}|H_1, V_2\rangle + |V_1, H_2\rangle \tag{24}$$

Hence, the singlet state is unchanged by the beam splitter, whereas the $|\Psi^+\rangle$ has both photons leaving the same output port. This is a manifestation that the singlet state has fermionic behavior (antisymmetric under exchange) and the other three states do not. Hence the triplets behave as bosons in which both photons leave the same output port.

Show that the $|\Phi^\pm\rangle$ cannot be distinguished after a beam splitter and that they behave as Bosons. Can they be distinguished from the other Bell states? What is the total amount of possible information that can then be sent in a "Dense Coding" protocol?